

Cracking and instability of isotropic and anisotropic relativistic spheres

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Abstract

Using the concept of cracking, we have explored the influence of density fluctuations on the stability of isotropic and anisotropic matter configurations in General Relativity with “barotropic” equations of state, $P = P(\rho)$ and $P_{\perp} = P_{\perp}(\rho)$. The concept of cracking, conceived to describe the behaviour of a fluid distribution just after its departure from equilibrium, provides an alternative and complementary approach to consider the stability of selfgravitating compact objects. We have refined the idea that density fluctuations affect other physical variables, but now including perturbation on radial pressure gradient and, the fact that perturbations must to be considered local, i.e. $\delta\rho = \delta\rho(r)$ and are represented by any function of compact support defined in a closed interval $\Delta r \ll 1$. It is found that not only anisotropic models could present cracking (or overturning), but also isotropic matter configurations could be affected by density fluctuation. We have also obtained that, under this method, previously unstable anisotropic models become stable.

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1 Introduction

Any compact object model is worthless if it is unstable against fluctuations of its physical variables. In General Relativity, stability of isotropic matter configurations has been extensively considered since the pioneer contributions of Bondi [1] and Chandrasekhar [2–4]. Bondi proposed a nice physically intuitive criterium of adiabatic stability while Chandrasekhar developed a detailed formalism that becomes a standard in General Relativity. Herrera and coworkers followed a qualitatively different approach and introduced the concept of cracking (or overturning) to study the stability of anisotropic relativistic fluids, just after its departure from equilibrium [5–7]. Under this cracking approach, a more recent contribution considered anisotropic fluids having two “barotropic” equations of state, $P = P(\rho)$ and $P_\perp = P_\perp(\rho)$, and studied only the effects of constant density perturbations on the stability of matter distributions [8]. These authors present a cracking criterion associated with the sign of the difference between the tangential and the radial sound speed, successfully applied to several astrophysical scenarios [9–11].

In this work we extend the idea that density fluctuations can affect other physical variables but now, we refine it assuming that non constant (local) perturbations can perturb the pressure gradient. The fluctuations are local, $\delta\rho = \delta\rho(r)$ and represented by any function of compact support defined in a closed interval $\Delta r \ll 1$. This paper is organized as follows: Section 2 will describe the concept of cracking for selfgravitating anisotropic matter configurations; in Section 3 we present the effects of local density fluctuations on the force distributions within isotropic/anisotropic matter configurations and we illustrate the effects with some examples and some conclusions are displayed in Section 4.

2 Cracking of anisotropic relativistic spheres

In a series of papers Herrera and coworkers [5–7] introduced the concepts of cracking and overturning to describe the behaviour of selfgravitating anisotropic matter configurations -just after its departure from equilibrium- when the radial force reverses its sign beyond some value of the radial coordinate within the configuration. They considered an static spherically symmetric metric,

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

and an anisotropic fluid,

$$T_{\mu\nu} = (\rho + P_\perp)u_\mu u_\nu - P_\perp g_{\mu\nu} + (P - P_\perp)v_\mu v_\nu, \quad (2)$$

where $u_\mu = (e^\nu, 0, 0, 0)$, $v_\mu = (0, -e^\lambda, 0, 0)$, ρ describes the energy density, P the radial pressure, and P_\perp the tangential pressure of the fluid.

Within this scenario, these authors showed [6, 7] that

$$\mathcal{R} = \frac{dP}{dr} + (\rho + P) \frac{m + 4\pi r^3 P}{r(r - 2m)} - \frac{2(P_\perp - P)}{r}, \quad (3)$$

evaluated at the moment immediately after perturbation, leads to

$$\mathcal{R} = -\frac{e^{2\lambda}(\rho + P)}{e^\nu r^2} \int_0^r d\tilde{r} e^\nu \tilde{r}^2 \frac{d\Theta}{ds} \quad (4)$$

where Θ represents the expansion. Observe that (3) is the hydrostatic equilibrium equation which vanishes for static (or quasi static evolving) configurations. It can be easily appreciated from (4) that cracking (or overturning) occurs at some value of $0 < r_c < R$ (where R denotes the boundary of the distribution), when \mathcal{R} -and consequently $d\Theta/ds$ - change its sign within the configuration. Cracking appears when the perturbed force distribution change its sign, $\delta\mathcal{R} < 0 \rightarrow \delta\mathcal{R} > 0$, i.e. at the inner part of the sphere the force pointing inward, reverses its sign at a given point r_c . Overturning occurs when the net force, directed outward, inverts its sign as $\delta\mathcal{R} > 0 \rightarrow \delta\mathcal{R} < 0$ at r_o .

3 The effects of local density perturbations

In this section we show the consequences of density perturbations $\delta\rho$ on the stability of an anisotropic matter configuration with two barotropic equations of state, $P = P(\rho)$ and $P_\perp = P_\perp(\rho)$. We assume that density fluctuations induce variations into all other physical variables, i.e. $m(r)$, $P(r)$ and $P_\perp(r)$, and their derivatives; affect the hydrostatic equilibrium of the system and lead to a non-vanishing total radial force distribution ($\delta\mathcal{R} \neq 0$) within the distribution.

3.1 Local density perturbations

First we note that a perturbation of density,

$$\rho \rightarrow \rho + \delta\rho, \quad (5)$$

induces a corresponding perturbation of their gradient,

$$\rho'(\rho + \delta\rho) \approx \rho'(\rho) + \delta\rho' = \rho'(\rho) + \frac{d\rho'}{d\rho}\delta\rho, \quad (6)$$

where primes stand for radial differentiation, which must be consistent with the expression

$$\frac{d}{dr} [\rho + \delta\rho] = \rho'(\rho) + \delta\rho' = \rho'(r) + \frac{d}{dr}\delta\rho, \quad (7)$$

in such a way that

$$\delta\rho' = \frac{d\rho'}{d\rho}\delta\rho = \frac{d}{dr}\delta\rho. \quad (8)$$

To have a consistent perturbation schema, we must to consider local perturbations of density, that can be properly described by any function of compact support, $\delta\rho = \delta\rho(r)$, defined in a closed interval $\Delta r \ll 1$.

Accordingly, these local density perturbations generate fluctuations in mass, radial pressure, tangential pressure and pressure gradient, that can be represented as

$$P(\rho + \delta\rho) \approx P(\rho) + \delta P = P(\rho) + \frac{dP}{d\rho} \delta\rho = P(\rho) + v^2 \delta\rho, \quad (9)$$

$$P_{\perp}(\rho + \delta\rho) \approx P_{\perp}(\rho) + \delta P_{\perp} = P_{\perp}(\rho) + \frac{dP_{\perp}}{d\rho} \delta\rho = P_{\perp}(\rho) + v_{\perp}^2 \delta\rho, \quad (10)$$

$$P'(\rho + \delta\rho) \approx P'(\rho) + \delta P' = P'(\rho) + \frac{dP'}{d\rho} \delta\rho = P'(\rho) + \left[(v^2)' + v^2 \frac{\rho''}{\rho'} \right] \delta\rho \quad (11)$$

and

$$m(\rho + \delta\rho) \approx m(\rho) + \delta m = m(\rho) + \frac{dm}{d\rho} \delta\rho = m(\rho) + \frac{4\pi r^2 \rho}{\rho'} \delta\rho, \quad (12)$$

where

$$v^2 = \frac{dP}{d\rho}, \quad (13)$$

$$v_{\perp}^2 = \frac{dP_{\perp}}{d\rho}, \quad (14)$$

and

$$m = 4\pi \int_0^r \rho(\bar{r}) \bar{r}^2 d\bar{r}, \quad (15)$$

denote the radial sound speed, the tangential sound speed, and mass, respectively. Notice that our perturbation scenario differs from the original one presented in [5–7], where fluctuations in density and anisotropy were considered independent and simultaneous. It also departs from the other previous study [8] because our approach assumes non constant density fluctuations which affects the pressure gradient, i.e. $\rho + \delta\rho(r) \rightarrow P'(\rho) + \delta P'$.

Following the above guidelines, we formally expand equation (3) as

$$\mathcal{R} \approx \mathcal{R}_0(\rho, P, P_{\perp}, P', m) + \delta\mathcal{R}, \quad (16)$$

where

$$\delta\mathcal{R} = \frac{\partial\mathcal{R}}{\partial\rho} \delta\rho + \frac{\partial\mathcal{R}}{\partial P} \delta P + \frac{\partial\mathcal{R}}{\partial P_{\perp}} \delta P_{\perp} + \frac{\partial\mathcal{R}}{\partial P'} \delta P' + \frac{\partial\mathcal{R}}{\partial m} \delta m, \quad (17)$$

and by using (5) we obtain

$$\delta\mathcal{R} = \delta\rho \left\{ \frac{\partial\mathcal{R}}{\partial\rho} + \frac{\partial\mathcal{R}}{\partial P} v^2 + \frac{\partial\mathcal{R}}{\partial P_{\perp}} v_{\perp}^2 + \frac{\partial\mathcal{R}}{\partial m} \left[\frac{4\pi r^2 \rho}{\rho'} \right] + \frac{\partial\mathcal{R}}{\partial P'} \left[(v^2)' + v^2 \frac{\rho''}{\rho'} \right] \right\}, \quad (18)$$

where the definition $\rho = \frac{m'}{4\pi r^2}$ has been considered. As can be easily calculated from (3), the derivatives of the force distribution, \mathcal{R} , are given by

$$\frac{\partial \mathcal{R}}{\partial \rho} = \frac{4\pi r^3 P + m}{r(r - 2m)}, \quad (19)$$

$$\frac{\partial \mathcal{R}}{\partial m} = \frac{(\rho + P)(1 + 8\pi r^2 P)}{(2m - r)^2}, \quad (20)$$

$$\frac{\partial \mathcal{R}}{\partial P} = \frac{m + 4\pi r^3(\rho + 2P)}{r(r - 2m)} + \frac{2}{r}, \quad (21)$$

$$\frac{\partial \mathcal{R}}{\partial P_\perp} = -\frac{2}{r} \quad (22)$$

$$(23)$$

and

$$\frac{\partial \mathcal{R}}{\partial P'} = 1. \quad (24)$$

To identify the possible sources of sign change, we can write (18) as

$$\delta \mathcal{R} = \delta \rho \left\{ \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \tilde{\mathcal{R}}_3 + \tilde{\mathcal{R}}_4 + \tilde{\mathcal{R}}_5 + \tilde{\mathcal{R}}_6 \right\}, \quad (25)$$

where

$$\tilde{\mathcal{R}}_1 = \frac{m + 4\pi r^3 P}{r(r - 2m)}, \quad (26)$$

$$\tilde{\mathcal{R}}_2 = \left[\frac{(\rho + P)(1 + 8\pi r^2 P)}{(r - 2m)^2} \right] \frac{4\pi}{3} r^3, \quad (27)$$

$$\tilde{\mathcal{R}}_3 = \left[\frac{m + 4\pi r^3(\rho + 2P)}{r(r - 2m)} \right] v^2, \quad (28)$$

$$\tilde{\mathcal{R}}_4 = 2 \left[\frac{v^2 - v_\perp^2}{r} \right], \quad (29)$$

$$\tilde{\mathcal{R}}_5 = (v^2)' + v^2 \frac{\rho''}{\rho'}, \quad (30)$$

and

$$\tilde{\mathcal{R}}_6 = \left[\frac{(\rho + P)(1 + 8\pi r^2 P)}{(2m - r)^2} \right] \left(\frac{\rho}{\rho'} - \frac{r}{3} \right) 4\pi r^2, \quad (31)$$

Observe that the expression for the perturbation of the net force (25) is independent of the arbitrary functional form of any density perturbation represented by a function of compact support defined in an interval $\Delta r \ll 1$ and notice the following facts:

1. any change of sign for $\delta \mathcal{R}$ should emerge from the last three terms because the first three are always be positive;

2. the contribution of the first four terms were considered in [8] for constant density perturbations. In the present work the last two terms emerge as a consequence of a variable density perturbation affecting both the pressure gradient and the mass function;
3. if the last three terms are negative but not big enough to counterbalance the first three, the configuration will not exhibit any cracking and will be stable under this type of density perturbations;
4. any perturbation at the center of the matter distribution (or at the surface $r = R$) should vanish in order that the spherical symmetry be preserved,

$$\delta\rho|_{r=0} \equiv 0 \quad \Rightarrow \quad \delta R|_{r=0} = 0. \quad (32)$$

Finally, it is worth mentioning that equation (25) admits cracking (or overturning) instabilities for isotropic matter distributions, when $P(r) = P_{\perp}(r)$, and can be written as

$$\delta\mathcal{R}_{\text{iso}} = \delta\rho \left\{ \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \tilde{\mathcal{R}}_3 + \tilde{\mathcal{R}}_5 + \tilde{\mathcal{R}}_6 \right\}, \quad (33)$$

as the term $\tilde{\mathcal{R}}_4$ vanishes.

3.2 Density perturbation on isotropic spheres

To study the effects of density perturbations on the stability of isotropic matter configurations, we examine a model proposed by Mehra [12] and considered physically viable by Delgaty and Lake [13]. The model is defined by density ρ and pressure P distributions written as

$$\rho = \frac{\rho_0(R^2 - r^2)(\sqrt{\alpha_1} + 2\sqrt{\alpha_2})^2}{R^2 \left[\alpha_1 + 4\sqrt{\alpha_1}\alpha_2 + 40\pi R^2\rho_0 - 40\pi R^2\rho_0 \cos\left(\frac{z-z_1}{2}\right)^2 \right]}, \quad (34)$$

$$P = \frac{1}{3} \frac{\rho_0 (R^2 - r^2)}{R^2}, \quad (35)$$

where

$$\alpha_1 = 225 - 240\pi a^2 \rho_0 \cos\left(\frac{z-z_1}{2}\right), \quad (36)$$

$$\alpha_2 = a\sqrt{10\pi\rho_0} \sin\left(\frac{z-z_1}{2}\right), \quad (37)$$

$$z = \ln\left(\frac{r^2}{R^2} - \frac{5}{6} + \sqrt{\frac{r^4}{R^4} - \frac{5r^2}{3R^2} + \frac{5}{8\pi R^2\rho_0}}\right), \quad (38)$$

and

$$z_1 = \ln\left(\frac{1}{6} + \sqrt{\frac{5}{8\pi R^2\rho_0} - \frac{2}{3}}\right), \quad (39)$$

with $R^2 < 9/10\pi\rho_0$. If $M = m(R)$ the central density, ρ_0 , can be written as

$$\rho_0 = \frac{15M}{8\pi R^3} = \frac{15\mu}{16\pi R^2}, \quad (40)$$

with $\mu = 2M/R$.

Figure 1 displays the force distribution, $\tilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the Mehra-model [12] with two different values of the mass-radius μ . Observe that the curve for $\mu = 0.2$ changes its sign around $r \approx 0.35$, which illustrates that a cracking instability can be found for this isotropic matter configurations.

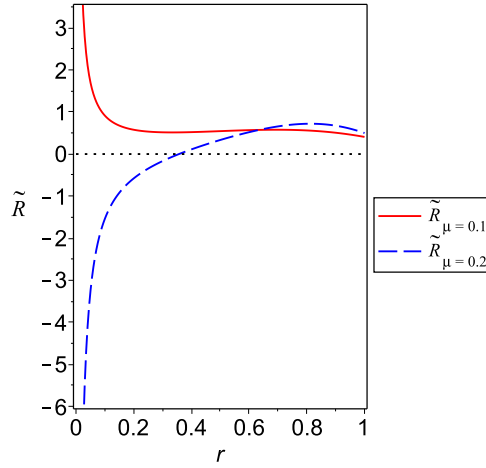


Figure 1: Force distribution, $\tilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the isotropic Mehra-Model [12] with $\mu = 0.1$ (solid line) and $\mu = 0.2$ (dashed line). The $\mu = 0.2$ curve presents a cracking point $r_c \approx 0.35$ which illustrates that cracking instability can be found for isotropic matter configurations.

3.3 Density perturbation on anisotropic spheres

In this section we present an analysis of two anisotropic models that were considered unstable in previous works. First, we will examine a model considered in [5], described by

$$\rho = \frac{K}{r^2}, \quad (41)$$

$$P = \frac{K}{3r^2} \left(\frac{7-9r^{-1/2}}{1-3r^{-1/2}} \right), \quad (42)$$

$$P_{\perp} = P - \frac{\gamma}{r^2}, \quad (43)$$

which proven to be unstable when simultaneous density and anisotropic perturbations take place. As it can be appreciated from Figure 2 the total distribution

force $\tilde{\mathcal{R}}$ does not change its sign, thus the model could be considered as potential stable under the present criterion (and also is under the previous sound velocity schema presented in [8]). The second model we study is based on the solution

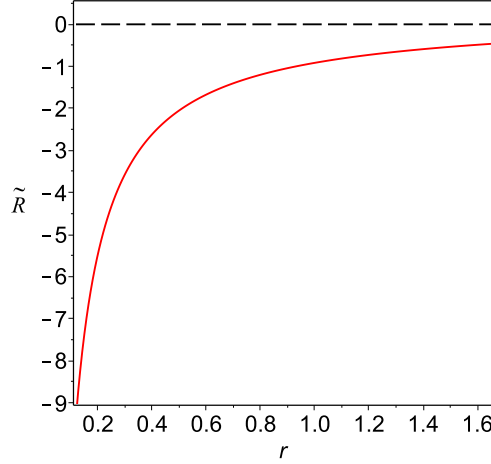


Figure 2: Force distribution, $\tilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the anisotropic Herrera-Model [5]. $\tilde{\mathcal{R}}$ does not change its sign and the model could be considered as potential stable. This picture differs from the one presented in [5].

derived by Gokhroo and Mehra [14], originally found by Florides [15] and later by Stewart [16], which is described by

$$\rho = \rho_0 \left(1 - \frac{Kr^2}{R^2} \right), \quad (44)$$

$$P = \gamma\rho_0 \left(1 - \frac{2m}{r} \right) \left(1 - \frac{r^2}{R^2} \right), \quad (45)$$

$$P_{\perp} = P + \frac{1}{2}rP' + \frac{(\rho + P)(m + 4\pi r^3 P)}{2(r - 2m)}, \quad (46)$$

with the central density ρ_0 written as

$$\rho_0 = \frac{15}{4\pi} \frac{M}{R^3(5 - 3K)} = \frac{15}{8\pi} \frac{\mu}{R^2(5 - 3K)}. \quad (47)$$

This model was studied in [8] and considered potentially unstable but, as it is clear from figure 3, with the present refinement of non-constant density perturbation - and assuming the same set of parameters: $\mu = 0.42$, $K = 3/56\pi$ and $\gamma = K/4$ - it does not present any cracking instability.

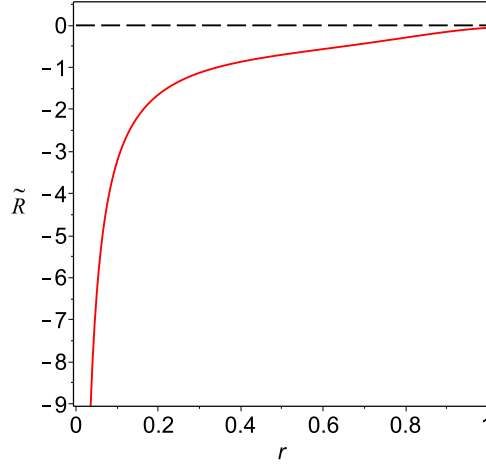


Figure 3: Force distribution, $\tilde{\mathcal{R}} = \delta\mathcal{R}/\delta\rho$, for the anisotropic Gokhroo/Mehra-Model [14] with: $\mu = 0.42$, $K = 3/56\pi$ and $\gamma = K/4$. Observe that it does not present any cracking instability reported in [8].

4 Remarks and conclusions

We have found that isotropic matter configurations present cracking (or overturning) when non constant (local) density fluctuations are considered. We have also obtained how the refinement of local density perturbations leads to stable models previously considered unstable. As we have mentioned, the perturbations we assume are local, $\delta\rho = \delta\rho(r)$ -represented by any function of compact support defined in a closed interval $\Delta r \ll 1$ - and affect all physical variables including the pressure gradient.

The idea of cracking was originally conceived by Herrera [5] to describe the behaviour of fluid distributions just after their departure from equilibrium: fluid elements, at both sides of the cracking point are accelerated with respect to each other by -independent and simultaneous- perturbations in energy density and anisotropy [6, 7]. In this approach, independent and simultaneous perturbations, may drive anisotropic matter configurations to exhibit cracking (or overturning). Later, Abreu, Hernández and Núñez [8] shown how constant (global) density perturbations could generate cracking on anisotropic relativistic fluids. In this study, constant density fluctuations affect mass, radial and tangential pressure, but leave unperturbed the pressure gradient; again only anisotropic distributions can exhibit cracking or overturning.

We extend this density-driven-perturbation approach assuming non-constant local density perturbations, which affect the gradient of pressure and lead to the possible presence of cracking instabilities for isotropic matter distributions. We

expect that perturbations occur on a confined neighbourhood nearby a particular point within the distribution. Thus, it is more realistic to assume non constant localised fluctuations -described by continuous non-zero functions around a breaking point- affecting the gradient of the pressure. The perturbation of the pressure gradient is also a suitable supposition because this gradient represents the distribution of forces within an hydrostatic configuration and should change with density fluctuations. Both extensions have proven to provide interesting outcomes because, we have obtained that isotropic matter distributions can also present cracking (or overturning) instability and, as it is shown, previous unstable models become stable under the present criterion.

It is worth to be mentioned that, the concept of cracking is complementary to the Bondi [1] and Chandrasekhar [2–4] stability criteria and it refers only to the tendency of the configuration to split (or to compress) at a particular point within the distribution but not to collapse or to expand. The cracking, overturning, expansion or collapse, has to be established from the integration of the full set of Einstein equations. Nevertheless, it should be clear that the occurrence of these phenomena could drastically alter the subsequent evolution of the system. If within a particular configuration no cracking (or overturning) is present, we could identify it as *potentially* stable (not absolutely stable), because other types of perturbations could lead to its expansions or collapse. Within a relativistic matter configuration, in principle, it is not clear which of the two scenarios is more likely to occur: the simultaneous two-perturbation original scenario of Herrera and coworkers or the only density-driven framework but surely both generate instabilities on relativistic matter configurations that should be evaluated.

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